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ISOPERIMETRIC ESTIMATES OF THE SOLUTIONS OF A CLASS OF PSEUDODIFFERENTIAL EQUATIONS AND THEIR APPLICATION TO CRACK PROBLEMS*

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Isoperimetric estimates are obtained of solutions of boundary value problems for a class of pseudodifferential equations. This class of equations includes the equation of problems on plane normal discontinuity cracks located in a homogeneous linearly elastic space and an inhomogeneous space whose Young's modulus has a power-law dependence on the distance to the plane of the crack. As it applies to crack problems, the established inequalities yield, in particular, isoperimetric estimates of the maximum opening of the crack and its volume under arbitrary loads.

1. Before formulating and proving the results obtained, let us recall the fundamental definitions and theorems to be utilized below.

The functions $f(\mathbf{x})$ and $g(\mathbf{x})$ are called equally measurable if $\forall a, b, \mu\{\mathbf{x}: a < f(\mathbf{x}) < b\} = \mu\{\mathbf{x}: a < g(\mathbf{x}) < b\}$. Here $\mu\{\ldots\}$ is a measure of the appropriate set.

It is said that the functions $f(\mathbf{x})$ and $g(\mathbf{x})$ have an identical direction of growth if $\forall \mathbf{x}, \mathbf{y}$, $(f(\mathbf{x}) - f(\mathbf{y}))(g(\mathbf{x}) - g(\mathbf{y})) \geqslant 0$.

Let the functions $f(\mathbf{x}) \geqslant 0$. We say that the function $f_*(\mathbf{x})$ is obtained from $f(\mathbf{x})$ by Schwartz symmetrization if $f_*(\mathbf{x})$ is equally measurable with $f(\mathbf{x})$ is spherically-symmetric, and does not increase as the radius increases.

Let $f(\mathbf{x}) \geqslant 0$, $g(\mathbf{x}) \geqslant 0$. Let us assume that $f_+(\mathbf{x})$ is equally measurable with $f(\mathbf{x})$ and $g_+(\mathbf{x})$ with $g(\mathbf{x})$, where the functions $f_+(\mathbf{x})$ and $g_+(\mathbf{x})$ have an identical direction of growth. Then the following inequality holds /1/:

$$\int f(\mathbf{x}) g(\mathbf{x})\, d\mathbf{x} \leqslant \int f_+(\mathbf{x}) g_+(\mathbf{x})\, d\mathbf{x} \tag{1.1}$$

Let $f(\mathbf{x}) \geqslant 0$, $g(\mathbf{x}) \geqslant 0$, $h(\mathbf{x}) \geqslant 0$, then /2, 3/

$$\int f(\mathbf{x}) g(\mathbf{x}-\mathbf{y}) h(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \leqslant \int f_*(\mathbf{x}) g_*(\mathbf{x}-\mathbf{y}) h_*(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \tag{1.2}$$

The main methods of constructing isoperimetric inequalities for solving differential equations were developed in /1/. In particular, the following estimate is proved in /1/.

Let $u_G(\mathbf{x})$ be a solution of the equation

$$-\Delta u(\mathbf{x}) = 1, \quad \mathbf{x} \in G, \quad G \subset R^n, \quad u|_{\partial G} = 0 \tag{1.3}$$

Then the following inequality holds:

$$\int_G u_G(\mathbf{x})\, d\mathbf{x} \leqslant \int_K u_K(\mathbf{x})\, d\mathbf{x} \tag{1.4}$$

where K is a sphere whose volume equals the volume of the domain G.

The estimate can be extended in an obvious manner to the case of an arbitrary right-hand side in Poisson's equation.

Let $u_G(f, \mathbf{x})$ be a solution of the equation

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad f(\mathbf{x}) \geqslant 0, \quad \mathbf{x} \in G, \quad u|_{\partial G} = 0 \tag{1.5}$$

Let us use the notation

$$W_G(f) = \int_G u_G(f, \mathbf{x}) f(\mathbf{x})\, d\mathbf{x}$$

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The following estimate holds

$$W_G(f) \leqslant W_K(f_*) \tag{1.6}$$

Then estimates (1.4) and (1.6) were generalized to other classes of equations. The validity of (1.4) and (1.6) is proved in /4/ for solving pseudodifferential equations of the form

$$p_G \Lambda^{\alpha} u(\mathbf{x}) + t^2 u(\mathbf{x}) = f(\mathbf{x}), \quad f(\mathbf{x}) \geqslant 0$$
$$\mathbf{x} \in G, \quad u(\mathbf{x}) \in \overset{\circ}{H}_{\alpha/2}(G), \quad 0 < \alpha < 2 \tag{1.7}$$

Here Λ^{α} is a pseudodifferential operator with the symbol $|\xi|^{\alpha}$ and p_G is the contraction in the domain G.

Inequality (1.4) is generalized to a differential equation with variable coefficients in the highest derivatives in /5/. Stronger inequalities than (1.4) and (1.6) /6, 7/ are proved for the solutions of Eqs.(1.3) and (1.5)

$$(u_G)_*(f, \mathbf{x}) \leqslant u_K(f_*, \mathbf{x}) \tag{1.8}$$

For equations containing the lower terms

$$-\Delta u(\mathbf{x}) + t^2 u(\mathbf{x}) = f(\mathbf{x}), \quad f(\mathbf{x}) \geqslant 0, \quad \mathbf{x} \in G, \quad u|_{\partial G} = 0$$

a somewhat less strong inequality /7/ is proved

$$\int_{K_r} (u_G)_*(f, \mathbf{x})\, d\mathbf{x} \leqslant \int_{K_r} u_K(f_*, \mathbf{x})\, d\mathbf{x} \tag{1.9}$$

(K_r is a pshere with centre at the origin and radius r, $r \leqslant r(K)$, and $r(K)$ is the radius of the sphere K).

The following inequalities

$$\max_{\mathbf{x}} u_G(f, \mathbf{x}) \leqslant \max_{\mathbf{x}} u_K(f_*, \mathbf{x}) \tag{1.10}$$

$$\int_G u_G(f, \mathbf{x})\, d\mathbf{x} \leqslant \int_K u_K(f_*, \mathbf{x})\, d\mathbf{x} \tag{1.11}$$

in particular, follow from (1.9).

The inequalities (1.9)-(1.11) are proved below for the solutions of (1.7). For simplicity we later set $t = 0$. The case $t > 0$ is examined without any changes.

2. We now denote by $u_G(f, \mathbf{x})$ the solution of the equation

$$p_G \Lambda^{\alpha} u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in G, \quad f(\mathbf{x}) \geqslant 0$$
$$u(\mathbf{x}) \in \overset{\circ}{H}_{\alpha/2}(G), \quad 0 < \alpha < 2 \tag{2.1}$$

As we know /4/, $u_G(f, \mathbf{x})$ is a minimum of the functional

$$I(G, f, u) = (\Lambda^{\alpha} u, u)/(f, u)^2, \quad u(\mathbf{x}) \in \overset{\circ}{H}_{\alpha/2}(G) \tag{2.2}$$

where $I(G, f, u_G(f, \mathbf{x})) = W_G^{-1}(f)$.

We first prove the inequalities obtained earlier /4/ by a different method and we set up the properties of the solutions.

Lemma 1. Let $u(\mathbf{x}) \in \overset{\circ}{H}_{\alpha/2}(G)$ and $u(\mathbf{x}) \geqslant 0$, then the following inequality holds:

$$(\Lambda^{\alpha} u, u) \geqslant (\Lambda^{\alpha} u_*, u_*) \tag{2.3}$$

Proof. According to /8/

$$(\Lambda^{\alpha} u, u) = C_{\alpha} \int \frac{|u(\mathbf{x}+\mathbf{y}) - u(\mathbf{x})|^2}{|\mathbf{y}|^{n+\alpha}}\, d\mathbf{x}\, d\mathbf{y} =$$
$$C_{\alpha} \int \frac{|u(\mathbf{z}) - u(\mathbf{x})|^2}{|\mathbf{z} - \mathbf{x}|^{n+\alpha}}\, d\mathbf{x}\, d\mathbf{z}, \quad C_{\alpha} = \text{const} \tag{2.4}$$

We let $K_m(s)$ denote a function defined in the domain $0 \leqslant s < \infty$

$$K_m(s) = \begin{cases} 1/s^{n+\alpha}, & s \geqslant 1/m \\ m^{n+\alpha}, & s < 1/m \end{cases}$$

We have

$$(\Lambda^\alpha u, u) = C_\alpha \lim_{m\to\infty} \int K_m(|\mathbf{z}-\mathbf{x}|)\,|u(\mathbf{z})-u(\mathbf{x})|^2\,d\mathbf{x}\,d\mathbf{z} = \lim_{m\to\infty} J_m(u)$$

$$J_m(u) = C_\alpha\left[\int u^2(\mathbf{z}) K_m(|\mathbf{z}-\mathbf{x}|)\,d\mathbf{x}\,d\mathbf{z} + \int u^2(\mathbf{x}) K_m(|\mathbf{z}-\mathbf{x}|)\,d\mathbf{x}\,d\mathbf{z} - 2S_m(u,u)\right] =$$
$$C_m \int u^2(\mathbf{x})\,d\mathbf{x} - 2C_\alpha S_m(u,u), \quad S_m(u,v) = \int u(\mathbf{z})\,v(\mathbf{x})\,K_m(|\mathbf{z}-\mathbf{x}|)\,d\mathbf{z}\,d\mathbf{x}$$

The first terms in the expresions for $J_m(u)$ and $J_m(u_*)$ are in agreement since the Schwartz symmetrization conserves L_2 as the norm of the functions. The inequality $S_m(u,u) \leqslant S_m(u_*, u_*)$ holds because of the above-mentioned inequality (1.2). We hence obtain $J_m(u) \geqslant J_m(u_*)$ and by passing to the limit as $m\to\infty$ we confirm the lemma (2.3). The inequality (2.3) was proved /4/ by using the theory of interpolation spaces.

The estimate (1.6) follows from the assertion in Lemma 1 and (2.2). Indeed

$$W_G^{-1}(f) = \frac{(\Lambda^\alpha u_G(f,\mathbf{x}), u_G(f,\mathbf{x}))}{(f, u_G(f,\mathbf{x}))} \geqslant \frac{(\Lambda^\alpha (u_G)_*(f,\mathbf{x}), (u_G)_*(f,\mathbf{x}))}{(f_*, (u_G)_*(f,\mathbf{x}))} \geqslant W_K^{-1}(f_*) \tag{2.5}$$

In (2.5) we used the fact that, by virtue of /9/, if $f \geqslant 0$ in (2.5) then $u_G(f,\mathbf{x}) \geqslant 0$ and we are in the conditions of Lemma 1. Moreover, the functions $f_*(\mathbf{x})$ and $(u_G)_*(f,\mathbf{x})$ have an identical direction of growth and by virtue of (1.1)

$$\int f(\mathbf{x})\,u_G(f,\mathbf{x})\,d\mathbf{x} \leqslant \int f_*(\mathbf{x})\,(u_G)_*(f,\mathbf{x})\,d\mathbf{x}$$

Note that it follows from the proof of (2.5) that the function $u_K(f_*, \mathbf{x})$ is invariant under Schwartz symmetrization, since otherwise Schwartz symmetrization could be applied to $u_K(f_*, \mathbf{x})$ by diminishing the value of the functionals (2.2).

3. We will now prove the inequalities (1.9)-(1.11) for the solutions of Eqs.(2.1).

Lemma 2. Let $u(\mathbf{x}), v(\mathbf{x}) \in \mathring{H}_{\alpha/2}(G)$, then

$$(\Lambda^\alpha u, v) = C_\alpha \int \frac{(u(\mathbf{x}+\mathbf{y}) - u(\mathbf{x}))\,\overline{(v(\mathbf{x}+\mathbf{y}) - v(\mathbf{x}))}}{|y|^{n+\alpha}}\,d\mathbf{x}\,d\mathbf{y} = \tag{3.1}$$
$$C_\alpha \int \frac{(u(\mathbf{z}) - u(\mathbf{x}))\,\overline{(v(\mathbf{z}) - v(\mathbf{x}))}}{|\mathbf{z}-\mathbf{x}|^{n+\alpha}}\,d\mathbf{z}\,d\mathbf{x}$$

The relationship (3.1) is proved in exactly the same way as the expression for $(\Lambda^\alpha u, u)$ is deduced in /8/.

We use the notation

$$u^\vee(\xi) = \int u(\mathbf{x})\,e^{i(\mathbf{x},\xi)}\,d\mathbf{x}$$

It follows from the relation

$$u(\mathbf{x}+\mathbf{y}) - u(\mathbf{x}) = \frac{1}{(2\pi)^n}\int u^\vee(\xi)\,(e^{-i(\mathbf{y},\xi)} - 1)\,e^{-i(\mathbf{x},\xi)}\,d\xi$$

analogous to the equality for $v(\mathbf{x})$ and the Parseval equality, that

$$\int \frac{(u(\mathbf{x}+\mathbf{y}) - u(\mathbf{x}))\,\overline{(v(\mathbf{x}+\mathbf{y}) - v(\mathbf{x}))}}{|\mathbf{y}|^{n+\alpha}}\,d\mathbf{x}\,dy = \tag{3.2}$$
$$\frac{1}{(2\pi)^n}\int \frac{u^\vee(\xi)\,\overline{v^\vee(\xi)}\,|e^{-i(\mathbf{y},\xi)} - 1|^2}{|\mathbf{y}|^{n+\alpha}}\,d\xi\,dy$$

According to /8/

$$\int \frac{|e^{-i(\mathbf{y},\xi)} - 1|^2}{|\mathbf{y}|^{n+\alpha}}\,d\mathbf{y} = C_\alpha'\,|\xi|^\alpha \tag{3.3}$$

From (3.2) and (3.3), (3.1) follows. Lemma 2 is proved.

Central to the proof of inequalities (1.9)-(1.11) is the following theorem.

Theorem 1. Let $u(\mathbf{x}) \geqslant 0,\ v(\mathbf{x}) \geqslant 0 \in \overset{\circ}{H}_{\alpha/2}(G)$ be real valued functions that have an identical direction of growth. Then the following inequality holds:

$$(\Lambda^\alpha u, v) \geqslant (\Lambda^\alpha u_*, v_*) \tag{3.4}$$

Proof. We use the notation

$$R_m(u, v) = C_\alpha \int K_m(|\mathbf{z}-\mathbf{x}|)(u(\mathbf{z}) - u(\mathbf{x}))(v(\mathbf{z}) - v(\mathbf{x}))\, d\mathbf{z}\, d\mathbf{x}$$

where K_m is the kernel introduced above, then

$$(\Lambda^\alpha u, v) = \lim_{m\to\infty} R_m(u, v)$$

According to (3.1)

$$R_m(u, v) = 2C_\alpha \int u(\mathbf{z}) v(\mathbf{z}) K_m(|\mathbf{z}-\mathbf{x}|)\, d\mathbf{x}\, d\mathbf{z} - 2C_\alpha S_m(u, v)$$

Integrating the right component with respect to x we obtain

$$R_m(u, v) = C_m \int u(\mathbf{z}) v(\mathbf{z})\, d\mathbf{z} - 2C_\alpha S_m(u, v)$$

We have an analogous equality on replacing $u,\ v$ by $u_*,\ v_*$.

Some the functions $u(\mathbf{x})$ and $v(\mathbf{x})$ have an identical direction of growth according to the assumption, while the functions $u_*(\mathbf{x}), v_*(\mathbf{x})$ are equally measurable with them and have an identical direction of growth by construction, then $\int u(\mathbf{z})\, v(\mathbf{z})\, d\mathbf{z} = \int u_*(\mathbf{z})\, v_*(\mathbf{z})\, d\mathbf{z}$. Moreover, $S_m(u, v) \leqslant S_m(u_*, v_*)$ by virtue of (1.2), Therefore, $R_m(u, v) \geqslant R_m(u_*, v_*)$. Passing to the limit here as $m \to \infty$, we arrive at (3.4). Theorem 1 is proved.

Theorem 2. The inequality

$$(\Lambda^\alpha (u_G)_*(f, \mathbf{x}), v_*(\mathbf{x})) \leqslant (\Lambda^\alpha u_K(f_*, \mathbf{x}), v_*(\mathbf{x})) \tag{3.5}$$

holds for the solutions of (2.1).

Here $v_*(\mathbf{x})$ is an arbitrary non-negative function, invariant under Schwartz symmetrization, from the space $\overset{\circ}{H}_{\alpha/2}(K)$.

Proof. Let the function $v(\mathbf{x})$ be equally measurable with $v_*(\mathbf{x})$ and have an identical direction of growth with the function $u_G(f, \mathbf{x})$. Then according to (3.4)

$$(\Lambda^\alpha u_G(f, \mathbf{x}), v(\mathbf{x})) \geqslant (\Lambda^\alpha (u_G)_*(f, \mathbf{x}), v_*(\mathbf{x})) \tag{3.6}$$

On the other hand

$$(\Lambda^\alpha u_G(f, \mathbf{x}), v(\mathbf{x})) = (f(\mathbf{x}), v(\mathbf{x})) \leqslant (f_*(\mathbf{x}), v_*(\mathbf{x})) = (\Lambda^\alpha u_K(f_*, \mathbf{x}), v_*(\mathbf{x})) \tag{3.7}$$

The derivation of inequalities (3.7) relies on the fact that the functions $f(\mathbf{x})$ and $f_*(\mathbf{x})$ as well as $v(\mathbf{x})$ and $v_*(\mathbf{x})$ are equally measurable, where $f_*(\mathbf{x})$ and $v_*(\mathbf{x})$ have an identical direction of growth. From (3.6) and (3.7) we obtain (3.5). Theorem 2 is proved.

We obtain the inequalities (1.9)-(1.11) for solutions of (2.1) as a corollary of Theorem 2.

We will prove the validity of (1.9). We examine the function $v_*(\mathbf{x})$ that is a solution of (2.1) in a sphere K with right-hand side $g_*(\mathbf{x})$ that equals one in the sphere K_r lying in K and is zero outside K_r. Since such a function $g_*(\mathbf{x})$ is invariant under Schwartz symmetrization, $v_*(\mathbf{x})$ possesses the same property as was mentioned above. Consequently, according to (3.5)

$$(\Lambda^\alpha (u_G)_*(f, \mathbf{x}), v_*(\mathbf{x})) = ((u_G)_*(f, \mathbf{x}), \Lambda^\alpha v_*(\mathbf{x})) = \int_{K_r} (u_G)_*(f, \mathbf{x})\, d\mathbf{x} \leqslant$$

$$(\Lambda^\alpha u_K(f_*, \mathbf{x}), v_*(\mathbf{x})) = \int_{K_r} u_K(f_*, \mathbf{x})\, d\mathbf{x}$$

The inequality (1.9), and therefore, the inequalities (1.10) and (1.11) are therefore proved.

For $n = 2$, Eqs.(2.1) correspond to problems concerning cracks in a homogeneous linearly elastic space $(\alpha = 1)$ and in an inhomogeneous space whose Young's modulus depends, as a power law on the distance to the plane of the crack $(E = E_\alpha |x_3|^{1-\alpha},\ 0 < \alpha < 1$ (it is assumed that the crack is located in the plane $x_3 = 0$). Eq.(1.7) corresponds to the problem of a crack between whose surfaces there are linear connections. The right-hand side of the equations $f(\mathbf{x})$ corresponds to applied forces, and the solution corresponds to the crack opening. Therefore, inequalities (1.9)-(1.11) yield estimates of the maximum opening and volume of a crack in the problems mentioned in terms of analogous characteristics of the problem of a circular crack with loads, obtained by using Schwartz symmetrization from the original load.

REFERENCES

1. POLYA G. and SZEGO G., Isoperimetric Inequalities in Mathematical Physics, Fizmatgiz, Moscow. 1962.
2. BRASCAMP H.J., LIEB E.H. and LUTTINGER J.M., A general rearrangement inequality for multiple integrals, J. Funct. Analysis, 17, 2, 1974.
3. LIEB E.H., Existence and uniqueness of the minimizing solution of Shoquard's non-linear equation, Stud. Appl. Math., 57, 2, 1977.
4. SHIFRIN E.I., Estimates of the solution of problems of a plane normal discontinuity crack in a material with power-law hardening, Izv. Akad. Nauk Arm SSR, Mekhanika, 37, 4, 1984.
5. BRUDNYI S.R. and SHIFRIN E.I., A method of symmetrizing functions and its application to certain elasticity theory problems for inhomogeneous bodies, PMM, 52, 3, 1988.
6. TALENTI G., Elliptic equations and rearrangements, Ann. Scuola Norm. Super. Pisa Ser. 3, 30, 4, 1976.
7. LIONS P.L., Quelques remarques sur la symmetrisation de Schwartz, Non-linear Partial Differential Equations and their Applications, Res. Notes in Math. 53, College de France Seminar, 1, Pitman, Boston, 1980.
8. ESKIN G.I., Boundary Value Problems for Elliptic Pseudodifferential Equations. Nauka, Moscow, 1973.
9. GOL'DSHTEIN R.V. and SHIFRIN E.I., Comparison theorems for a certain class of pseudodifferential equations and their application, Dokl. Akad. Nauk SSSR, 262, 5, 1982.

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